Acta Appl Math (2017) 148:179–199
DOI 10.1007/s10440-016-0084-z

Soliton Solutions for a Singular Schrödinger Equation with Any Growth Exponents

Jiayin Liu[1,2] · Duchao Liu[2] · Peihao Zhao[2]

Received: 15 January 2015 / Accepted: 21 November 2016 / Published online: 21 December 2016

Abstract This paper is concerned with a kind of quasilinear Schrödinger equation with combined nonlinearities, a convex term with any growth and a singular term, in a bounded smooth domain. Multiplicity results are obtained by critical point theory together with truncation arguments and the method of upper and lower solutions.

Keywords Quasilinear Schrödinger equation · Singular equation · Critical or supercritical exponent · Variational methods · Moser iteration technique

Mathematics Subject Classification 35D05 · 35J20 · 35J75

1 Introduction and Main Results

Consider the following quasilinear Schrödinger equation

$$\begin{cases} -\Delta_p u - \frac{p}{2^{p-1}} u \Delta_p(u^2) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

J. Liu supported by the Scientic Research Project for Colleges and Universities in Ningxia Hui Autonomous Region (No. NGY2016135) and the Research Starting Funds for Imported Talents of Beifang University of Nationalities.
D. Liu supported by Fundamental Research Funds for Central Universities (lzujbky-2014-25).
P. Zhao supported by the NSF of China (NSFC11471147).

✉ J. Liu
xecd@163.com

D. Liu
liudch@lzu.edu.cn

P. Zhao
zhaoph@lzu.edu.cn

1 School of Mathematics and Information Science, Beifang University of Nationalities, Yinchuan 750021, People's Republic of China

2 School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, People's Republic of China

where Ω is a domain in $\mathbb{R}^N$ $(N \geq 3)$. Quasilinear equations of form (1.1), referred as so-called Modified Nonlinear Schrödinger Equation due to the quasilinear and non-convex term $u\Delta_p u^2$, and have been derived as model of several physical phenomena (see [4–7, 19, 20, 29]).

Problems of type (1.1) were studied primarily in the context of $p = 2$ and subcritical case. In this connection, we refer the readers to [9, 14, 22, 24, 26, 27, 33, 41]. One may note that one of the main difficulties of the quasilinear problem (1.1) is that there is no suitable space on which the energy functional is well defined and belongs to C^1 class. To overcome this difficulty, several ideas and techniques were developed, including the constrained minimization argument [14, 24, 33], the Nehari manifold [27], the method of a change of variables [9, 26] and the perturbation method [22]. For critical case, extra difficulties arise since the lack of compactness of the Sobolev embedding. As pointed by Liu et al. in [26], the critical case for (1.1) is very interesting. Concerning this case, Moameni in [31] considers the related singularly perturbed problem and obtains a positive radial solution in the radially symmetric case. Later on, an existence result of positive solutions was given by João Marcos et al. in [12] via Mountain-Pass Theorem and P.L. Lions' Concentration-Compactness Principle. Recently, Liu et al. consider the existence of positive solutions for a quasilinear elliptic equation like (1.1) in [23] by perturbation method. Deng et al. [13] and Liu et al. [25] deal with a general type of elliptic equation like (1.1) and obtain a positive solution by variational argument and P.L. Lions' Concentration-Compactness Principle. However, there seems to be little progress on the existence of positive solution for (1.1) with critical growth. Up to now, to the authors' best knowledge, there is no one considering problem (1.1) with supercritical nonlinearities.

Recently, there appeared some works dealing with (1.1) when $p \neq 2$. For example, Liu [21] and Liu and Zhao [28] consider problem (1.1) in a bounded smooth domain, to our best knowledge, this is the only results established for the p-Laplacian case in a bounded domain.

Motivated by above results, in this paper, we consider the p-Laplacian case in a bounded smooth domain, but this time, different from above works, our perturbations involving a singular term, i.e. we consider the following quasilinear Schrödinger equation

$$\begin{cases} -\Delta_p u - \frac{p}{2^{p-1}} u\Delta_p(u^2) = \lambda u^\beta + a(x)u^{-\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded smooth domain, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $1 < p < \infty$, $\lambda > 0$ is a parameter, γ and β are positive constants. $a(x) \geq 0$ is a nontrivial measurable function. To emphasize the dependence on λ or β, problem (P) is often referred to as (P_λ) or $(P_{\lambda,\beta})$, and the subscript λ or β is omitted if no confusion arises.

As mentioned before, a major difficulty associated with (P) is the following: one may seek to obtain solutions by looking for critical points of the associated "natural" functional: $J(u) : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$J(u) := \frac{1}{p}\int_\Omega \big(1 + p|u|^p\big)|\nabla u|^p \mathrm{d}x - \frac{\lambda}{\beta+1}\int_\Omega g(u)^{\beta+1}\mathrm{d}x - \frac{1}{1-\gamma}\int_\Omega a(x)g(u)^{1-\gamma}.$$

However, this functional is not well-defined for all $u \in W_0^{1,p}(\Omega)$, hence it is difficult to apply variational methods directly. To overcome this difficulty, we use the method of changing variables developed in [9, 26] for $p=2$, [21, 28] for p-Laplacian case (i.e. $1<p<\infty$), and make a new different definition of weak solutions. That is

$$v := g^{-1}(u),$$

where g is defined by

$$\begin{aligned} g'(t) &= \frac{1}{(1+p|g(t)|^p)^{1/p}}, \quad \forall t \in [0,+\infty), \\ g(t) &= -g(-t), \quad \forall t \in (-\infty, 0]. \end{aligned} \tag{1.2}$$

We now make use of a change of unknown $v = g^{-1}(u)$ (note that g is smooth and invertible, see Lemma 2.1 for more information about g), and define an associated equation

$$\begin{cases} -\Delta_p v = [\lambda g(v)^\beta + a(x) g(v)^{-\gamma}] g'(v) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_N}$$

It is easy to see that problem (P_N) is equivalent to our problem (P), which takes $u = g(v)$ as its solutions. More precisely, we say u is a weak solution of (P), if $v = g^{-1}(u) \in W_0^{1,p}(\Omega)$ is a positive weak solution of problem (P_N), i.e.

$$\int_\Omega |\nabla v|^{p-2} \nabla v \nabla \varphi \mathrm{d}x = \lambda \int_\Omega g(v)^\beta g'(v) \varphi \mathrm{d}x + \int_\Omega a(x) g(v)^{-\gamma} g'(v) \varphi \mathrm{d}x$$

for every $\varphi \in W_0^{1,p}(\Omega)$.

Let $p^* := \frac{Np}{N-p}$ (respectively $+\infty$) if $p < N$ (respectively $p \geq N$). We introduce the following assumption on the function $a(x)$.

(H) There are $\varphi_0 \geq 0$ in $C_0^1(\overline{\Omega})$ and $q > N$ such that $a(x)\varphi_0^{-\gamma} \in L^q(\Omega)$.

Now we can state our main results as follows

Theorem 1.1 *Suppose (H) holds, let $1 < p < \frac{\beta+1}{2} < p^*$ and $\gamma > 0$. Then there exists $\lambda_* > 0$ such that for all $\lambda \in (0, \lambda_*)$, problem (P) has two solutions.*

Theorem 1.2 *Suppose (H) holds, let $\frac{\beta+1}{2} \geq p^*$ and $\gamma > 0$. Then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$, problem (P) has two solutions.*

Remark 1.3 Note that we do not require that $\gamma < 1$, a restriction that appears often in the literature (see, e.g., Sun et al. [34], Wang et al. [35], Zhang [40]).

Remark 1.4 Note that $\beta + 1 = 2p^*$ behaves like a critical exponent for (P_N) (see Lemma 2.1, property (8) in next section). Compared with the works mentioned before, in the case of $\beta + 1 = 2p^*$ or $\beta + 1 > 2p^*$, problem (P_N) becomes more complicated since the effects of the singular term and the nonlinearity g and the loss of compactness of Sobolev embedding.

Recently, there appeared some works dealing with singular equations driven by p-Laplacian, we mention the works of Agarwal et al. [1], Zhao et al. [39], Perera and Zhang [32], Gasínski and Papageorgiou [17], Giacomoni et al. [18]. In those mentioned works, the authors deal with subcritical nonlinearities. In the case of critical or supercritical exponent ($\beta + 1 \geq 2p^*$), which will be studied in Sect. 3, problem (P_N) becomes more delicate because the Sobolev embedding $W_0^{1,p}(\Omega) \subset L^{\frac{\beta+1}{2}}(\Omega)$ is not compact. To our best knowledge, there is no one considering p-Laplacian with both such singularities and critical or supercritical nonlinearities. In this paper, We will deal with critical case and supercritical case in a unified approach by using truncation techniques and Moser iteration technique, which are different from the methods used in [12, 13, 31]. For Laplacian equation with a singular term and critical nonlinearities, we refer the reader to [35, 38]. In [35], the existence of positive solutions have been proved by Nehari manifold and P.L. Lions' Concentration-Compactness Principle, while in [38], the multiplicity results have been obtained by Ekeland variational principle and careful analysis the minimax level for which the compactness can be established.

This paper is organized as follows. In Sect. 2, we consider the subcritical case (i.e. $\beta + 1 < 2p^*$) and give the proof of Theorem 1.1; In Sect. 3, we deal with our problem in critical case and supercritical case (i.e. $\beta + 1 \geq 2p^*$) in a unified approach and give the proof of Theorem 1.2.

Throughout this paper, we make use of the following notation. $L^p(\Omega)$, $1 \leq p \leq \infty$, denotes Lebesgue space; the norm in $L^p(\Omega)$ is denoted by $\|\cdot\|_p$; the norm in $W_0^{1,p}(\Omega)$ is denoted by $\|\cdot\|$; $C, C_0, C_1, C_2, \dots$ denote (possibly different) positive constants; X^* denotes the dual space of Banach space X, $(W_0^{1,p}(\Omega))^*$ is $W^{-1,p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$; λ_1 denotes the first eigenvalue of $-\Delta_p$ with zero Dirichlet condition on Ω, $\widehat{v}_1 > 0$ is the eigenfunction corresponding to λ_1 with $\|\widehat{v}_1\|_p = 1$.

2 Subcritical Case: $\beta + 1 < 2p^*$

As we have mentioned in previous section, $\beta + 1 = 2p^*$ behaves like a critical exponent for problem (P_N), this can be seen from properties (8) in Lemma 2.1 below. In this section, we consider the subcritical case $\beta + 1 < 2p^*$. We point out that since (P_N) is equivalent to problem (P), so, we only consider problem (P_N) in this and the following sections.

Next, let us summarize some properties of the function g defined by (1.2). For its proof, we refer to [21, 28].

Lemma 2.1 *The function g defined by* (1.2) *satisfies the following conditions*:

(1) $g(0) = 0$;
(2) *g is uniquely defined, C^∞ and invertible*;
(3) $0 < g'(t) \leq 1$ *for all* $t \in \mathbb{R}$;
(4) $\frac{1}{2} g(t) \leq t g'(t) \leq g(t)$ *for all* $t > 0$;
(5) $g(t)/t \nearrow 1$, *as* $t \to 0+$;
(6) $|g(t)| \leq |t|$ *for all* $t \in \mathbb{R}$;
(7) $g(t)/\sqrt{t} \nearrow K_0 := \sqrt{2} p^{-1/(2p)}$, *as* $t \to +\infty$;
(8) $|g(t)| \leq K_0 |t|^{1/2}$ *for all* $t \in \mathbb{R}$;
(9) $g^2(t) - g(t) g'(t) t \geq 0$ *for all* $t \in \mathbb{R}$;
(10) *There exists a positive constant C such that* $|g(t)| \geq C|t|$ *for* $|t| \leq 1$ *and* $|g(t)| \geq C|t|^{1/2}$ *for* $|t| \geq 1$;
(11) $|g(t) g'(t)| < K_0^2$ *for all* $t \in \mathbb{R}$;
(12) $g''(t) < 0$ *when* $t > 0$ *and* $g''(t) > 0$ *when* $t < 0$.

Lemma 2.2 [8, Proposition 17.3] *Let* $1 < p < +\infty$. *There exist two positive constants* c_p *and* C_p *such that for every* $\xi,\ \eta \in \mathbb{R}^N$

$$c_p N_p(\xi,\eta) \le \left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)\cdot(\xi-\eta) \le C_p N_p(\xi,\eta),$$

a dot denotes here the Euclidean scalar product in $\mathbb{R}^N$, *and* $N_p(\xi,\eta) := \{|\xi| + |\eta|\}^{p-2}|\xi - \eta|^2$.

Lemma 2.3 [32, Proposition 2.1] *Suppose* $h \in L^q(\Omega)$ *for some* $q > N$. *Then the Dirichlet problem*

$$\begin{cases} -\Delta_p u = h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique solution $u \in C_0^1(\overline{\Omega})$. *Moreover if* $h \ge 0$ *is nontrivial, then*

$$u > 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} > 0 \quad \text{on } \partial\Omega,$$

where ν *is the interior unit normal on* $\partial\Omega$.

Let $A : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\left\langle A(u), y\right\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla y \mathrm{d}x \quad \text{for all } u, y \in W_0^{1,p}(\Omega). \tag{2.1}$$

Lemma 2.4 [16] *The map* $A : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)^*$ *defined by* (2.1) *is type of* (S_+), *that is, if* $u_n \rightharpoonup u$ *in* $W_0^{1,p}(\Omega)$ *and* $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \le 0$, *then* $u_n \to u$ *in* $W_0^{1,p}(\Omega)$.

Since we will use upper-lower solution method to produce the first solution, so, we recall the definitions of lower solution and upper solution of problem (P_N).

Definition 2.5 We say $v \in W_0^{1,p}(\Omega)$ is a weak lower solution (weak upper solution) of the boundary value problem (P_N) if

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla\varphi \mathrm{d}x \le (\ge)\lambda \int_\Omega g(v)^\beta g'(v)\varphi \mathrm{d}x + \int_\Omega a(x) g(v)^{-\gamma} g'(v)\varphi \mathrm{d}x$$

and

$$u \le (\ge) 0 \quad \text{on } \partial\Omega$$

for every $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \ge 0$.

Next, we generate lower and upper solutions for problem (P_N). First, we produce a lower solution.

Lemma 2.6 *Assume hypothesis* (*H*) *holds, then problem* (P_N) *admits a positive lower solution* $\underline{v} \in C_0^1(\bar{\Omega})$.

Proof Consider the following Dirichlet problem:

$$\begin{cases} -\Delta_p v = a(x)g'(v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Since hypothesis (H) implies that $a(x) \in L^q(\Omega), q > N$, note that $0 < g'(v) < 1$, by Lemma 2.3, problem (2.2) has a positive solution $\underline{\omega} \in C_0^1(\bar{\Omega})$, and $\underline{\omega} > 0$ in Ω. Fix $0 < t \leq 1$ so small that $\underline{v}(x) = t\underline{\omega}(x) \in [0, 1)$, $\forall x \in \bar{\Omega}$. Then using (2.2) and the fact that $a(x) \geq 0$, $\underline{v}(x) \in (0, 1)$ for all $x \in \Omega$, we have

$$\begin{aligned} -\Delta_p \underline{v} &= t^{p-1} a(x) g'(\underline{v}) \\ &\leq a(x) g'(\underline{v}) \\ &\leq a(x) g'(\underline{v}) \underline{v}^{-\gamma} \\ &\leq a(x) g'(\underline{v}) g(\underline{v})^{-\gamma} \\ &\leq \lambda g(\underline{v})^{\beta} g'(\underline{v}) + a(x) g'(\underline{v}) g(\underline{v})^{-\gamma}. \end{aligned}$$

So $\underline{v} \in C_0^1(\bar{\Omega})$ is a lower solution for problem (P_N). □

Next, we produce an upper solution for (P_N).

Lemma 2.7 *If hypothesis* (*H*) *holds, then there exists* $\lambda_0 > 0$ *such that , for all* $\lambda \in (0, \lambda_0)$, *problem* ($P_N$) *has an upper solution* $\overline{v} \in C_0^1(\overline{\Omega})$. *Moreover,* $\overline{v} \geq \underline{v}$ *in* $\overline{\Omega}$.

Proof Consider the problem

$$\begin{cases} -\Delta_p v = a(x) g'(\underline{v}) g(\underline{v})^{-\gamma} + 1 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Hypothesis (H) implies $a(x) g'(\underline{v}) g(\underline{v})^{-\gamma} \in L^q(\Omega), q > N$, by Lemma 2.3, problem (2.3) has a solution $\overline{v} \in C_0^1(\overline{\Omega})$,

$$\begin{aligned} -\Delta_p \overline{v} &= a(x) g'(\underline{v}) g(\underline{v})^{-\gamma} + 1 \\ &\geq a(x) g'(\underline{v}) g(\underline{v})^{-\gamma} \\ &\geq -\Delta_p \underline{v}. \end{aligned}$$

So, $\overline{v} \geq \underline{v}$. It follows that

$$\begin{aligned} -\Delta_p \overline{v} - a(x) g'(\overline{v}) g(\overline{v})^{-\gamma} - \lambda g(\overline{v})^{\beta} g'(\overline{v}) &\geq -\Delta_p \overline{v} - a(x) g'(\underline{v}) g(\underline{v})^{-\gamma} - \lambda g(\overline{v})^{\beta} g'(\overline{v}) \\ &= 1 - \lambda g(\overline{v})^{\beta} g'(\overline{v}) \\ &\geq 1 - \lambda \overline{v}^{\beta}, \end{aligned}$$

where we have used $g(\overline{v})^{-\gamma} \leq g(\underline{v})^{-\gamma}$, $g'(\overline{v}) \leq g'(\underline{v})$, since g is increasing and g' is non-increasing (see Lemma 2.1). So, there exists $\lambda_0 > 0$ such that $1 - \lambda \overline{v}^{\beta} \geq 0$ for all $\lambda < \lambda_0$. Hence, $\overline{v}$ is an upper solution of (P_N) for $\lambda \in (0, \lambda_0)$. □

Using $\underline{v}$, $\overline{v}$ obtained by Lemma 2.6 and Lemma 2.7, truncation techniques and direct variational methods, we will prove the following existence result.

Lemma 2.8 *Suppose* (*H*) *holds, let* $0 < \beta < +\infty$, $\gamma > 0$, *then for all* $\lambda \in (0, \lambda_0)$, *problem* (P_N) *has a solution*

$$v_0 \in C_0^1(\overline{\Omega}) \quad \textit{with } \underline{v} \le v_0 \le \overline{v} \quad \textit{in } \overline{\Omega}.$$

Proof Let

$$f(x,\zeta) = \begin{cases} [\lambda g(\underline{v})^{\beta} + a(x) g(\underline{v})^{-\gamma}] g'(\underline{v}), & \zeta < \underline{v}, \\ [\lambda g(\zeta)^{\beta} + a(x) g(\zeta)^{-\gamma}] g'(\zeta), & \underline{v} \le \zeta \le \overline{v}, \\ [\lambda g(\overline{v})^{\beta} + a(x) g(\overline{v})^{-\gamma}] g'(\overline{v}), & \zeta > \overline{v}. \end{cases} \tag{2.4}$$

Evidently $f(x,\zeta)$ is a Carathéodory function. We set

$$F(z,\zeta) = \int_0^{\zeta} f(x,s)\mathrm{d}s$$

and consider the C^1-functional $\hat{I} : W_0^{1,p}(\Omega) \to \mathbb{R}$, defined by

$$\hat{I}(v) = \frac{1}{p}\int_{\Omega} |\nabla v|^p \mathrm{d}x - \int_{\Omega} F(x,v)\mathrm{d}x, \ \forall v \in W_0^{1,p}(\Omega).$$

From (2.4), it is clear that $\hat{I}$ is coercive. Moreover, exploiting the compactness of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, we can easily verify that $\hat{I}$ is sequentially weakly lower semicontinuous. Hence by the Weierstrass theorem, we can find $v_0 \in W_0^{1,p}(\Omega)$, such that

$$\hat{I}(v_0) = \inf\big\{\hat{I}(v) : v \in W_0^{1,p}(\Omega)\big\}. \tag{2.5}$$

So, we have

$$\hat{I}'(v_0) = 0$$

i.e.

$$\int_{\Omega} |\nabla v_0|^{p-2} \nabla v_0 \nabla \varphi \mathrm{d}x = \int_{\Omega} f(x,v_0)\varphi \mathrm{d}x, \quad \forall \varphi \in W_0^{1,p}(\Omega). \tag{2.6}$$

Taking $\varphi = (v_0 - \overline{v})^+ \in W_0^{1,p}(\Omega)$ as test function in (2.6), using (2.4) and Lemma 2.7, we obtain

$$\begin{aligned} \int_{\Omega} |\nabla v_0|^{p-2} \nabla v_0 \nabla (v_0 - \overline{v})^+ \mathrm{d}x &= \int_{\Omega} f(x,v_0)(v_0 - \overline{v})^+ \mathrm{d}x \\ &= \int_{\{v_0 > \overline{v}\}} f(x,v_0)(v_0 - \overline{v})\mathrm{d}x \\ &= \int_{\{v_0 > \overline{v}\}} \big[\lambda g(\overline{v})^{\beta} + a(x) g(\overline{v})^{-\gamma}\big] g'(\overline{v})(v_0 - \overline{v})\mathrm{d}x \end{aligned}$$

$$\leq \int_{\Omega} \left[\lambda g(\overline{v})^{\beta} + a(x) g(\overline{v})^{-\gamma}\right] g'(\overline{v})(v_0 - \overline{v})^{+} \mathrm{d}x$$

$$\leq \int_{\Omega} |\nabla \overline{v}|^{p-2} \nabla \overline{v} \nabla (v_0 - \overline{v})^{+} \mathrm{d}x,$$

where $\{v_0 > \overline{v}\}$ denotes the set defined by $\{v_0 > \overline{v}\} = \{x \in \Omega \mid v_0(x) > \overline{v}(x)\}$. This means

$$\int_{\Omega} \left\{ |\nabla v_0|^{p-2} \nabla v_0 - |\nabla \overline{v}|^{p-2} \nabla \overline{v} \right\} \cdot \nabla (v_0 - \overline{v})^{+} \mathrm{d}x \leq 0.$$

From Lemma 2.2 it follows that for some constant $c_p > 0$,

$$c_p \int_{\{v_0 > \overline{v}\}} \left(|\nabla v_0|^{p-2} + |\nabla \overline{v}|^{p-2}\right) |\nabla (v_0 - \overline{v})|^2 \mathrm{d}x \leq 0.$$

It follows that

$$\nabla (v_0 - \overline{v})^{+} = 0 \quad \text{in } \Omega,$$

and by the Poincaré inequality that $(v_0 - \overline{v})^{+} = 0$ in Ω, i.e.

$$v_0 \leq \overline{v}. \tag{2.7}$$

In a similar fashion, acting on (2.6) with $\varphi = (\underline{v} - v_0)^{+} \in W_0^{1,p}(\Omega)$ and using this time Lemma 2.6, we obtain

$$\underline{v} \leq v_0. \tag{2.8}$$

From (2.7) and (2.8), we obtain

$$\underline{v} \leq v_0 \leq \overline{v}.$$

Lemma 2.3 yields $v_0 \in C_0^1(\overline{\Omega})$. Using similar arguments as in [2, 32], we have if

$$\|v - v_0\|_{C^1} = \varepsilon$$

with ε small, then $\underline{v} \leq v \leq \overline{v}$. Moreover $I(v) - \widetilde{I}(v)$ is constant for $\underline{v} \leq v \leq \overline{v}$, and therefore v_0 is also a local minimum of $I(v)$ in the C^1-topology. Now, we invoke [15, Theorem 1.1] to claim that v_0 is also a local minimum of $I(v)$ in the $W_0^{1,p}(\Omega)$ topology. Thus

$$\begin{cases} -\Delta_p v_0 = [\lambda g(v_0)^{\beta} + a(x) g(v_0)^{-\gamma}] g'(v_0) & \text{in } \Omega, \\ v_0 > 0 & \text{in } \Omega, \\ v_0 = 0 & \text{on } \partial\Omega \end{cases}$$

and it is a solution of (P_N). This completes the proof. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1 Let v_0 be the first solution produced by Lemma 2.8, then using v_0, we introduce the following Carathéodory function:

$$g_0(x,\zeta)=\begin{cases}[\lambda g(v_0)^{\beta}+a(x)g(v_0)^{-\gamma}]g'(v_0), & \zeta\le v_0,\\ [\lambda g(\zeta)^{\beta}+a(x)g(\zeta)^{-\gamma}]g'(\zeta), & \zeta> v_0,\end{cases} \tag{2.9}$$

and consider the problem

$$\begin{cases}-\Delta_p v=g_0(x,v) & \text{in } \Omega,\\ v_0=0 & \text{on } \partial\Omega.\end{cases} \tag{2.10}$$

Every solution of problem (2.10) is bigger than v_0 and hence also a solution of (P_N). Note that solutions of (2.10) are the critical points of the C^1 functional $I_0 : W_0^{1,p}(\Omega)\to\mathbb{R}$, defined by

$$I_0(v)=\frac{1}{p}\int_{\Omega}|\nabla v|^p\mathrm{d}x-\int_{\Omega}G_0(x,v)dx,$$

where $G_0(x,\zeta)=\int_0^{\zeta}g_0(x,s)\mathrm{d}s$.

Claim 1. I_0 satisfies the C-condition.

Let $\{v_n\}\subseteq W_0^{1,p}(\Omega)$ be a sequence such that

$$|I_0(v_n)|\le C_1 \tag{2.11}$$

for some constant $C_1>0$ and

$$(1+\|v_n\|)I_0'(v_n)\to 0 \quad \text{in } W^{-1,p'}(\Omega).$$

This is equivalent to

$$\left|\int_{\Omega}|\nabla v_n|^{p-2}\nabla v_n\nabla h\mathrm{d}x-\int_{\Omega}g_0(x,v_n)h\mathrm{d}x\right|\le\frac{\varepsilon_n\|h\|}{1+\|v_n\|},\ \forall h\in W_0^{1,p}(\Omega), \tag{2.12}$$

with $\varepsilon_n\to 0$. Choosing $h=v_n\in W_0^{1,p}(\Omega)$ in the above inequality, we have

$$\int_{\Omega}g_0(x,v_n)v_n\mathrm{d}x-\int_{\Omega}|\nabla v_n|^p\mathrm{d}x\le\varepsilon_n. \tag{2.13}$$

On the other hand, from (2.11), we have

$$-\int_{\Omega}pG_0(x,v_n)\mathrm{d}x+\int_{\Omega}|\nabla v_n|^p\mathrm{d}x\le C_2 \tag{2.14}$$

for some constant $C_2>0$. Adding (2.13) and (2.14), we obtain

$$\int_{\Omega}\big(g_0(x,v_n)v_n-pG_0(x,v_n)\big)\mathrm{d}x\le C_3 \tag{2.15}$$

for some constant $C_3>0$. Denote

$$\Pi=\int_{\{v_n\ge v_0\}}\big(g_0(x,v_n)v_n-pG_0(x,v_n)\big)\mathrm{d}x.$$

Hence, by (2.9), (2.15) and $\frac{1}{2}g(t) \le g'(t)t \le g(t)$ for all $t > 0$ (see Lemma 2.1), we have

$$\begin{aligned} C_3 &\ge \int_{\{v_n<v_0\}} \big(g_0(x,v_n)v_n - pG_0(x,v_n)\big)\mathrm{d}x + \Pi \\ &\ge (1-p)\int_{\{v_n<v_0\}} \big(\lambda g(v_0)^\beta + a(x)g(v_0)^{-\gamma}\big)g'(v_0)v_n \mathrm{d}x + \Pi \\ &\ge (1-p)\int_{\{v_n<v_0\}} \big(\lambda g(v_0)^\beta + a(x)g(v_0)^{-\gamma}\big)g'(v_0)v_0 \mathrm{d}x + \Pi \\ &\ge (1-p)\int_{\{v_n<v_0\}} \big(\lambda g(v_0)^{1+\beta} + a(x)g(v_0)^{1-\gamma}\big)\mathrm{d}x + \Pi. \end{aligned}$$

Since $p > 1$, we have

$$C_4 \ge \int_{\{v_n\ge v_0\}} \big(g_0(x,v_n)v_n - pG_0(x,v_n)\big)\mathrm{d}x, \tag{2.16}$$

for some constant $C_4 > 0$. It is easy to see that, if $\gamma \ne 1$,

$$\begin{aligned} \Pi &= \int_{\{v_n\ge v_0\}} \big(g_0(x,v_n)v_n - pG_0(x,v_n)\big)\mathrm{d}x \\ &= \int_{\{v_n\ge v_0\}} \Big[\big(\lambda g(v_n)^\beta + a(x)g(v_n)^{-\gamma}\big)g'(v_n)v_n - p\big(\lambda g(v_0)^\beta + a(x)g(v_0)^{-\gamma}\big)g'(v_0)v_0 \\ &\quad - \frac{\lambda p}{1+\beta}\big(g(v_n)^{1+\beta} - g(v_0)^{1+\beta}\big) - \frac{p}{1-\gamma}a(x)\big(g(v_n)^{1-\gamma} - g(v_0)^{1-\gamma}\big)\Big]\mathrm{d}x \end{aligned} \tag{2.17}$$

if $\gamma = 1$,

$$\begin{aligned} \Pi &= \int_{\{v_n\ge v_0\}} \big(g_0(x,v_n)v_n - pG_0(x,v_n)\big)\mathrm{d}x \\ &= \int_{\{v_n\ge v_0\}} \Big[\big(\lambda g(v_n)^\beta + a(x)g(v_n)^{-\gamma}\big)g'(v_n)v_n - p\big(\lambda g(v_0)^\beta + a(x)g(v_0)^{-\gamma}\big)g'(v_0)v_0 \\ &\quad - \frac{\lambda p}{1+\beta}\big(g(v_n)^{1+\beta} - g(v_0)^{1+\beta}\big) - pa(x)\big(\ln g(v_n) - \ln g(v_0)\big)\Big]\mathrm{d}x. \end{aligned} \tag{2.18}$$

Next, we demonstrate that $\|v_n\| \le C$ for some positive constant C which is independent of n. Actually, may assume $v_n \ge 1$, for otherwise we are done by (2.14). If $\gamma > 1$, noting that $\frac{1}{2} > \frac{p}{1+\beta}$ and $\frac{1}{2}g(t) \le g'(t)t \le g(t)$ for all $t > 0$ (see Lemma 2.1), by (2.16) and (2.17), we obtain

$$\begin{aligned} C_5 &\ge \int_{\{v_n\ge v_0\}} \left(\frac{1}{2} - \frac{p}{1+\beta}\right)\lambda g(v_n)^{1+\beta}\mathrm{d}x \\ &\ge \int_{\{v_n\ge v_0\}} \left(\frac{1}{2} - \frac{p}{1+\beta}\right)\lambda v_n^{\frac{1+\beta}{2}}\mathrm{d}x \end{aligned}$$

for some constant $C_5 > 0$. Then

$$\int_\Omega v_n^{\frac{1+\beta}{2}}\mathrm{d}x \le C_6 \tag{2.19}$$

for some positive constant C_6. If $0 < \gamma < 1$, we can use Hölder inequality and Lemma 2.1 to obtain

$$\begin{aligned}\frac{p}{1-\gamma}\int_{\{v_n\geq v_0\}} a(x)g(v_n)^{1-\gamma}\mathrm{d}x &\leq C_7\int_{\{v_n\geq v_0\}} a(x)v_n^{\frac{1-\gamma}{2}}\mathrm{d}x\\ &\leq C_7|\Omega|^{1-\frac{1}{q}-\frac{1-\gamma}{1+\beta}}\|a\|_q\left(\int_{\{v_n\geq v_0\}} v_n^{\frac{1+\beta}{2}}\mathrm{d}x\right)^{\frac{1-\gamma}{1+\beta}}\\ &\leq C_8\left(\int_{\{v_n\geq v_0\}} v_n^{\frac{1+\beta}{2}}\mathrm{d}x\right)^{\frac{1-\gamma}{1+\beta}} \qquad (2.20)\end{aligned}$$

for some positive constants C_7 and C_8. Thus, it follows from (2.16), (2.17) and (2.20) that

$$\begin{aligned}C_9 &\geq \int_{\{v_n\geq v_0\}}\left(\frac{1}{2}-\frac{p}{1+\beta}\right)\lambda g(v_n)^{1+\beta}\mathrm{d}x - \frac{p}{1-\gamma}\int_{\{v_n\geq v_0\}} a(x)g(v_n)^{1-\gamma}\mathrm{d}x\\ &\geq C_{10}\int_{\{v_n\geq v_0\}} v_n^{\frac{1+\beta}{2}}\mathrm{d}x - C_8\left(\int_{\{v_n\geq v_0\}} v_n^{\frac{1+\beta}{2}}\mathrm{d}x\right)^{\frac{1-\gamma}{1+\beta}}\end{aligned}$$

for some positive constant C_9 and C_{10}. Noting that $0 < \frac{1-\gamma}{1+\beta} < 1$, we have

$$\int_{\{v_n\geq v_0\}} v_n^{\frac{1+\beta}{2}}\mathrm{d}x \leq C_{11},$$

and hence we obtain

$$\int_{\Omega} v_n^{\frac{1+\beta}{2}}\mathrm{d}x \leq C_{12} \qquad (2.21)$$

for some positive constants C_{11} and C_{12}. If $\gamma = 1$, by (2.16) and (2.18), we have

$$C_{13} \geq \int_{\{v_n\geq v_0\}}\left(\frac{1}{2}-\frac{p}{1+\beta}\right)\lambda g(v_n)^{1+\beta}\mathrm{d}x - p\int_{\{v_n\geq v_0\}} a(x)\big[\ln g(v_n) - \ln g(v_0)\big]\mathrm{d}x, \qquad (2.22)$$

for some constant $C_{13} > 0$. From Lemma 2.1, we know $\frac{1}{2t} \leq \frac{g'(t)}{g(t)} \leq \frac{1}{t}$ for $t > 0$, combing this fact with mean value theorem, there exists $\delta \in (0, 1)$ such that

$$\begin{aligned}\ln g(v_n) - \ln g(v_0) &= \frac{g'(v_0+\delta(v_n-v_0))}{g(v_0+\delta(v_n-v_0))}(v_n-v_0)\\ &\leq \frac{v_n-v_0}{v_0+\delta(v_n-v_0)} \leq \frac{1}{\delta},\end{aligned}$$

and (2.22) yields

$$\int_{\Omega} v_n^{\frac{1+\beta}{2}}\mathrm{d}x \leq C_{14} \qquad (2.23)$$

for some constant $C_{14} > 0$. From (2.19), (2.21) and (2.23), we have, for all $\gamma > 0$,

$$\int_{\Omega} v_n^{\frac{1+\beta}{2}}\mathrm{d}x \leq C_{15} \qquad (2.24)$$

for some positive constant C_{15}. Returning to (2.14) and using (2.9) and the fact that $g'(t)g(t) < g(t),\ \forall t > 0$, we have

$$\begin{aligned}\int_\Omega |\nabla v_n|^p \mathrm{d}x &\le C_2 + \int_\Omega pG_0(x, v_n)\mathrm{d}x \\ &= C_2 + \int_{\{v_n < v_0\}} pG_0(x, v_n)\mathrm{d}x + \int_{\{v_n \ge v_0\}} pG_0(x, v_n)\mathrm{d}x \\ &\le C_{16} + p\int_{\{v_n \ge v_0\}} G_0(x, v_n)\mathrm{d}x. \end{aligned} \tag{2.25}$$

Using (2.24) and (2.25) (Similarly as before, we distinguish three cases: $\gamma > 1$, $0 < \gamma < 1$ and $\gamma = 1$, the details are omitted), we have

$$\int_\Omega |\nabla v_n|^p \mathrm{d}x \le C_{18} + \int_\Omega v_n^{\frac{1+\beta}{2}} \mathrm{d}x \le C$$

for some constant $C > 0$. Therefore, we may assume that

$$\begin{aligned} v_n &\rightharpoonup v \quad L^p(\Omega), \\ v_n &\rightharpoonup v \quad W_0^{1,p}(\Omega). \end{aligned}$$

In (2.12), we choose $h = v_n - v \in W_0^{1,p}(\Omega)$, then

$$\left| \int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla(v_n - v)\mathrm{d}x - \int_\Omega g_0(x, v_n)(v_n - v)\mathrm{d}x \right| \le \frac{\varepsilon_n \|v_n - v\|}{1 + \|v_n\|}.$$

So $\int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla(v_n - v)\mathrm{d}x \to 0$, using Lemma 2.4, we have

$$v_n \to v \ \text{ in } \ W_0^{1,p}(\Omega).$$

This proves Claim 1.

Claim 2. $I_0(t\hat{v}_1) \to -\infty$ as $t \to \infty$.

Indeed, Using the fact $\|\nabla \hat{v}_1\|_p^p = \lambda_1 \|\hat{v}_1\|_p^p$ and (2.9), we obtain

$$\begin{aligned} I_0(t\hat{v}_1) &= \frac{t^p}{p}\int_\Omega |\nabla \hat{v}_1|^p \mathrm{d}x - \int_\Omega G_0(x, t\hat{v}_1)\mathrm{d}x \\ &\le \frac{\lambda_1 t^p}{p}\int_\Omega |\hat{v}_1|^p \mathrm{d}x - \int_{\{t\hat{v}_1 > v_0\}} G_0(x, t\hat{v}_1)\mathrm{d}x \\ &= \frac{\lambda_1}{p}\int_{\{t\hat{v}_1 \le v_0\}} (t\hat{v}_1)^p \mathrm{d}x + \int_{\{t\hat{v}_1 > v_0\}} \left(\frac{\lambda_1}{p}(t\hat{v}_1)^p - G_0(x, t\hat{v}_1)\right)\mathrm{d}x \\ &\le C_{11} + \int_{\{t\hat{v}_1 > v_0\}} \left(\frac{\lambda_1}{p}(t\hat{v}_1)^p - \frac{\lambda}{1+\beta}g(t\hat{v}_1)^{1+\beta} - \frac{1}{1-\gamma}a(x)g(t\hat{v}_1)^{1-\gamma}\right)\mathrm{d}x. \end{aligned}$$

Since $p < \frac{\beta+1}{2}$ and $g(t)$ behaves like $t^{1/2}$ for t large enough, we have

$$I_0(t\hat{v}_1) \to -\infty \quad \text{as } t \to +\infty.$$

This proves Claim 2.

Claim 3. We can find $\rho \in (0, 1)$ small enough, such that

$$I_0(v_0) < \inf\{I_0(v) : \|v - v_0\| = \rho\}$$

Define a Carathéodory function on $\Omega \times \mathbb{R}$ by

$$\widetilde{g}_0(x,\zeta) = \begin{cases} [\lambda g(v_0)^\beta + a(x)g(v_0)^{-\gamma}]g'(v_0), & \zeta < v_0, \\ [\lambda g(\zeta)^\beta + a(x)g(\zeta)^{-\gamma}]g'(\zeta), & v_0 \le \zeta \le \overline{v}. \\ [\lambda g(\overline{v})^\beta + a(x)g(\overline{v})^{-\gamma}]g'(\overline{v}), & \zeta > \overline{v}, \end{cases}$$

and consider the problem

$$\begin{cases} -\Delta_p v = \widetilde{g}_0(x,v) & \text{in } \Omega, \\ v_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.26}$$

The corresponding functional is

$$\widetilde{I}_0(v) = \frac{1}{p}\int_\Omega |\nabla v|^p \mathrm{d}x - \int_\Omega \widetilde{G}_0(x,v)dx,$$

where $\widetilde{G}_0(x,\zeta) = \int_0^\zeta \widetilde{g}_0(x,s)\mathrm{d}s$. Note that $v_0 \in C_0^1(\overline{\Omega})$ is a solution of (2.26), hence a lower solution of (2.26). Moreover, since $v_0 \le \overline{v}$, from (2.9), we see that $\overline{v}$ is still an upper solution of (2.26). By similar technique as the proof of Lemma 2.8, we assume $\widetilde{v}_0$ is the global minimizer of the functional $\widetilde{I}_0$. If $\widetilde{v}_0 \ne v_0$, we are done; If $\widetilde{v}_0 = v_0$, since $\widetilde{I}_0 = I_0$ in a $C_0^1(\overline{\Omega})$-neighborhood of v_0, so v_0 is a local minimum of I_0 in the $C_0^1(\overline{\Omega})$ topology, hence also a local minimum of I_0 in the $W_0^{1,p}(\Omega)$ topology, so Claim 3 holds.

Since Claim 1, Claim 2 and Claim 3 hold, then the Mountain Pass Theorem [3, 16, 36] now gives a second critical point $v_1 \in W_0^{1,p}(\Omega)$ for I_0, hence a solution of problem (P_N). This completes the proof. □

3 The Critical or Supercritical Case: $\beta + 1 \ge 2p^*$

In this section, we investigate the solvability of (P_N) in the case of critical or supercritical exponent. Since $\frac{1+\beta}{2} \ge p^*$, we point out that the nonlinearity $\lambda g(v)^\beta g'(v) + a(x)g(v)^{-\gamma}g'(v)$ has a critical or supercritical growth, and we can not use the variational techniques directly, by virtue of the lack of compactness of the Sobolev embedding. So, following the idea in [10, 30, 42], we construct a suitable truncation of $\lambda g(v)^\beta g'(v) + a(x)g(v)^{-\gamma}g'(v)$ in order to use variational methods.

Let $K > 0$ be a real number, whose value will be fixed later, and consider the functional $h_K : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$h_K(v) = \begin{cases} 0, & v \le 0, \\ g(v)^\beta g'(v), & 0 \le v \le K, \\ g(K)^{\beta-s}g(v)^s g'(v), & v \ge K, \end{cases} \tag{3.1}$$

where s is a positive constant satisfying $p < \frac{s+1}{2} < p^* \le \frac{\beta+1}{2}$. The function h_K enjoys the following conditions:

$$|h_K(v)| \le g(K)^{\beta-s}g(v)^s g'(v). \tag{3.2}$$

Next, we investigate the following truncated problem associated to h_K

$$\begin{cases} -\Delta_p v = \lambda h_K(v) + a(x)g(v)^{-\gamma}g'(v) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_K}$$

Since (H) holds and $p < \frac{s+1}{2} < p^*$, by Theorem 1.1, problem (P_K) has two positive solutions v_1, a local minimum, and v_2 is of mountain pass type. More precisely, $I_K(v_2) = c_M$, where c_M is the mountain pass level associated to the functional

$$I_K(v) = \frac{1}{p}\int_\Omega |\nabla v|^p \mathrm{d}x - \lambda \int_\Omega H_K(v)\mathrm{d}x - \frac{1}{1-\gamma}\int_\Omega a(x)g(v)^{1-\gamma}\mathrm{d}x$$

which is related to the problem (P_K), where

$$H_K(v) = \int_0^v h_K(t)dt. \tag{3.3}$$

Obviously, one has $I_K(v_i) \le m$ for some $m > c_M > 0$ independent of λ since v_i, $H_K(v_i)$ and $a(x)g(v_i)^{1-\gamma}$ are positive functions ($i = 1,\ 2$). To prove Theorem 1.2, we need the following estimate.

Lemma 3.1 *Let v_1 and v_2 are solutions of problem (P_K), then $\|v_i\| \le m_0,\ i = 1, 2,$ for all $\lambda \ge 0$, where $m_0 > 0$ is a constant does not depend on λ.*

Proof Let v_1 and v_2 be the solutions of (P_K). For simplicity, denote $v = v_i,\ i = 1, 2$, in this proof. Noting that $\frac{1}{2}g(t) \le tg'(t) \le g(t)$ for all $t > 0$ (see Lemma 2.1) and $s < \beta$, we can use (3.1) and (3.3) to deduce

$$\begin{aligned}
&\int_\Omega \left(H_K(v) - \frac{2}{1+s}h_K(v)v\right)dx \\
&= \int_{\{0\le v\le K\}} \left(\frac{1}{1+\beta}g(v)^{1+\beta} - \frac{2}{1+s}g(v)^\beta g'(v)v\right)\mathrm{d}x \\
&\quad + g(K)^{\beta-s}\int_{\{v\ge K\}} \left(\frac{1}{1+s}g(v)^{1+s} - \frac{2}{1+s}g(v)^s g'(v)v\right)\mathrm{d}x \\
&\le \int_{\{0\le v\le K\}} \left(\frac{1}{1+\beta}g(v)^{1+\beta} - \frac{1}{1+s}g(v)^{1+\beta}\right)\mathrm{d}x \\
&\le \int_{\{0\le v\le K\}} \left(\frac{1}{1+\beta} - \frac{1}{1+s}\right)g(v)^{1+\beta}\mathrm{d}x \\
&\le 0.
\end{aligned} \tag{3.4}$$

If $\gamma \ne 1$, using again $\frac{1}{2}g(t) \le tg'(t) \le g(t)$ for all $t > 0$ and (3.4), we have

$$\begin{aligned}
m \ge I_K(v) &= I_K(v) - \frac{2}{1+s}I_K'(v)v \\
&\ge \left(\frac{1}{p} - \frac{2}{1+s}\right)\int_\Omega |\nabla v|^p \mathrm{d}x - \frac{1}{1-\gamma}\int_\Omega a(x)g(v)^{1-\gamma}\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{1+s}\int_\Omega a(x)g(v)^{-\gamma}g'(v)v\mathrm{d}x\\
&\geq\left(\frac{1}{p}-\frac{2}{1+s}\right)\int_\Omega|\nabla v|^p\mathrm{d}x-\frac{1}{1-\gamma}\int_\Omega a(x)g(v)^{1-\gamma}\mathrm{d}x+\frac{1}{1+s}\int_\Omega a(x)g(v)^{1-\gamma}\mathrm{d}x\\
&\geq\left(\frac{1}{p}-\frac{2}{1+s}\right)\int_\Omega|\nabla v|^p\mathrm{d}x-\left(\frac{1}{1-\gamma}-\frac{1}{1+s}\right)\int_\Omega a(x)g(v)^{1-\gamma}\mathrm{d}x. \qquad (3.5)
\end{aligned}
$$

If $\gamma>1$, since $\frac{1}{1-\gamma}-\frac{1}{1+s}<0$, it follows from (3.5) that

$$
m\geq\left(\frac{1}{p}-\frac{2}{1+s}\right)\int_\Omega|\nabla v|^p\mathrm{d}x.
$$

Hence,

$$
\|v\|\leq C_{19} \qquad (3.6)
$$

for some positive constant C_{19} independent of λ.

If $0<\gamma<1$, by Hölder inequality, $g(t)\leq t$ and Sobolev embedding theorem, we get

$$
\begin{aligned}
\int_\Omega a(x)g(v)^{1-\gamma}\mathrm{d}x&\leq\int_\Omega a(x)v^{1-\gamma}\mathrm{d}x\\
&\leq\|a\|_q\|v\|_{p^*}^{1-\gamma}|\Omega|^{1-\frac{1}{q}-\frac{1-\gamma}{p^*}}\\
&\leq C_{20}\|v\|^{1-\gamma},
\end{aligned}
$$

for some constant $C_{20}>0$. Thus, by (3.5), we have

$$
m\geq\left(\frac{1}{p}-\frac{2}{1+s}\right)\|v\|^p-C_{20}\|v\|^{1-\gamma}.
$$

Then

$$
\|v\|\leq C_{21} \qquad (3.7)
$$

for some positive constant C_{21} independent of λ.

If $\gamma=1$, using Hölder inequality and Sobolev embedding theorem, we have

$$
\begin{aligned}
\int_\Omega a(x)v(x)\mathrm{d}x&\leq\|a\|_q\|v\|_{p^*}|\Omega|^{1-\frac{1}{q}-\frac{1}{p^*}}\\
&\leq C_{22}|\Omega|^{1-\frac{1}{q}-\frac{1}{p^*}}\|a\|_q\|v\|\\
&\leq C_{23}\|v\|,
\end{aligned}
$$

for some positive constant C_{22} and C_{23}. then (3.4) yields

$$
\begin{aligned}
m&\geq I_K(v)=I_K(v)-\frac{2}{1+s}I_K'(v)v\\
&\geq\left(\frac{1}{p}-\frac{2}{1+s}\right)\|v\|^p-\int_\Omega a(x)\ln g(v)\mathrm{d}x+\frac{2}{1+s}\int_\Omega a(x)\frac{g'(v)v}{g(v)}\mathrm{d}x
\end{aligned}
$$

$$\geq \left(\frac{1}{p} - \frac{2}{1+s}\right)\|v\|^p - \int_\Omega a(x)\ln g(v)\mathrm{d}x$$
$$\geq \left(\frac{1}{p} - \frac{2}{1+s}\right)\|v\|^p - \int_\Omega a(x)v(x)\mathrm{d}x$$
$$\geq \left(\frac{1}{p} - \frac{2}{1+s}\right)\|v\|^p - C_{23}\|v\|,$$

hence, we obtain

$$\|v\| \leq C_{24} \tag{3.8}$$

for some positive constant C_{24} independent of λ. It follows from (3.6), (3.7) and (3.8) that

$$\|v\| \leq m_0$$

for some constant $m_0 > 0$ does not depend on λ. The proof is completed. □

Remark 3.2 One should note that c_M is dependent on K, actually, c_M is decreasing with respect to K, so, we may assume m_0 is also decreasing with respect to K, this fact is important in the following $L^\infty(\Omega)$ estimate (see inequality (3.18) in the proof of Theorem (1.2)).

Indeed, v_1 and v_2 also solve problem (P_N), it reduces to an $L^\infty(\Omega)$ estimate, in other words, we only need to prove $\|u_i\|_{L^\infty(\Omega)} \leq K$, $(i = 1, 2)$ for some $K > 0$. Next, we are going to use Moser iteration method [10, 11, 30, 37, 42] to prove Theorem 1.2.

Proof of Theorem 1.2 For the sake of simplicity, we shall use the following notation:

$$v := v_i, \quad i = 1, 2.$$

For $L > 0$, let us define the following functions

$$v_L(x) = \begin{cases} v(x), & \text{if } v(x) \leq L, \\ L, & \text{if } v(x) > L, \end{cases}$$

$$z_L = v_L^{p(\tau-1)}(v-K)^+ \quad \text{and} \quad w_L = v_L^{\tau-1}v,$$

where $\tau > 1$ will be fixed later. Let us use z_L as a test function in (P_K), that is

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla z_L \mathrm{d}x = \lambda \int_\Omega h_K(v)g'(v)z_L\mathrm{d}x + \int_\Omega a(x)g(v)^{-\gamma}g'(v)z_L\mathrm{d}x. \tag{3.9}$$

Put $D := \{x \in \Omega : v(x) \geq K\}$. By Hölder inequality and (3.2), notice that $\frac{1}{2}g(t) \leq g'(t)t \leq g(t)$ for $t > 0$ and $|g(t)| \leq K_0|t|^{\frac{1}{2}}$ for all $t \in \mathbb{R}$ (see Lemma 2.1), we infer that the right hand side (RHS for short) of (3.9) is

$$RHS = \lambda \int_D h_K(v)z_L\mathrm{d}x + \int_D a(x)g(v)^{-\gamma}g'(v)z_L\mathrm{d}x$$
$$= \lambda \int_D h_K(v)v_L^{p(\tau-1)}(v-K)^+\mathrm{d}x + \int_D a(x)g(v)^{-\gamma}g'(v)v_L^{p(\tau-1)}(v-K)^+\mathrm{d}x$$

$$
\begin{aligned}
&\le \lambda g(K)^{\beta-s}\int_D g(v)^s g'(v) v_L^{p(\tau-1)}(v-K)\mathrm{d}x+\int_D a(x)g(v)^{-\gamma}g'(v)v_L^{p(\tau-1)}(v-K)\mathrm{d}x\\
&\le \lambda K^{\beta-s}\int_D g(v)^s g'(v) v v_L^{p(\tau-1)}\mathrm{d}x+\int_D a(x)g(v)^{-\gamma}g'(v) v v_L^{p(\tau-1)}\mathrm{d}x\\
&\le \lambda K^{\beta-s}\int_D g(v)^{s+1} v_L^{p(\tau-1)}\mathrm{d}x+\int_D a(x)g(v)^{1-\gamma} v_L^{p(\tau-1)}\mathrm{d}x\\
&\le \lambda K^{\beta-s}K_0^{s+1}\int_D v^{\frac{s+1}{2}} v_L^{p(\tau-1)}\mathrm{d}x+K_0^{1-\gamma}\int_D a(x)v^{\frac{1-\gamma}{2}} v_L^{p(\tau-1)}\mathrm{d}x\\
&\le \lambda K^{\beta-s}K_0^{s+1}\int_D v^{\frac{s+1}{2}-p} w_L^{p}\mathrm{d}x+K_0^{1-\gamma}\int_D a(x)v^{\frac{1-\gamma}{2}-p} w_L^{p}\mathrm{d}x\\
&\le \lambda K^{\beta-s}K_0^{s+1}\int_D v^{\frac{s+1}{2}-p} w_L^{p}\mathrm{d}x+K_0^{1-\gamma}K^{\frac{1-\gamma}{2}-p}\int_D a(x) w_L^{p}\mathrm{d}x\\
&\le \lambda K^{\beta-s}K_0^{s+1}\|v\|_{p^*}^{\frac{p^*(\alpha^*-p)}{\alpha^*}}\|w_L\|^p_{L^{\alpha^*}(D)}+K_0^{1-\gamma}K^{\frac{1-\gamma}{2}-p}|\Omega|^{\frac{1}{\theta}}\|a\|_q\|w_L\|^p_{L^{\alpha^*}(D)}\\
&\le \Big[\lambda K^{\beta-s}K_0^{s+1}\|v\|_{p^*}^{\frac{p^*(\alpha^*-p)}{\alpha^*}}+K_0^{1-\gamma}K^{\frac{1-\gamma}{2}-p}|\Omega|^{\frac{1}{\theta}}\|a\|_q\Big]\|w_L\|^p_{L^{\alpha^*}(D)},
\end{aligned}
\tag{3.10}
$$

where $\alpha^* := \frac{pp^*}{p^*-\frac{s+1}{2}+p}$, $\frac{1}{\theta}+\frac{1}{q}+\frac{p}{\alpha^*}=1$ (this choice of θ is reasonable since we can fix s such that $\frac{s+1}{2}<p^*$ but close to p^*). Returning to the left hand side (LHS for short) of (3.9), and using the definition of v_L, we obtain

$$
\begin{aligned}
LHS &= \int_\Omega |\nabla v|^{p-2}\nabla v\nabla z_L \mathrm{d}x\\
&= \int_D |\nabla v|^{p-2}\nabla v\nabla\big(v_L^{p(\tau-1)}(v-K)^+\big)\mathrm{d}x\\
&= \int_D |\nabla v|^{p-2}\nabla v\big(v_L^{p(\tau-1)}\nabla v+p(\tau-1)v_L^{p(\tau-1)-1}(v-K)\nabla v_L\big)\mathrm{d}x\\
&= \int_D |\nabla v|^{p} v_L^{p(\tau-1)}\mathrm{d}x+p(\tau-1)\int_{D\cap\{v\le L\}} v_L^{p(\tau-1)-1}(v-K)|\nabla v|^{p-2}\nabla v\nabla v_L\mathrm{d}x\\
&= \int_D |\nabla v|^{p} v_L^{p(\tau-1)}\mathrm{d}x+p(\tau-1)\int_{D\cap\{v\le L\}} v_L^{p(\tau-1)-1}(v-K)|\nabla v|^{p}\mathrm{d}x\\
&\ge \int_D |\nabla v|^{p} v_L^{p(\tau-1)}\mathrm{d}x.
\end{aligned}
\tag{3.11}
$$

From (3.9)–(3.11), we have

$$
\begin{aligned}
&\int_D |\nabla v|^{p} v_L^{p(\tau-1)}\mathrm{d}x\\
&\quad\le \Big[\lambda K^{\beta-s}K_0^{s+1}\|v\|_{p^*}^{\frac{p^*(\alpha^*-p)}{\alpha^*}}+K_0^{1-\gamma}K^{\frac{1-\gamma}{2}-p}|\Omega|^{\frac{1}{\theta}}\|a\|_q\Big]\|w_L\|^p_{L^{\alpha^*}(D)}.
\end{aligned}
\tag{3.12}
$$

Since $\tau > 1$, by Sobolev embedding theorem, we get

$$\begin{aligned}\left(\int_D |w_L|^{p^*}\mathrm{d}x\right)^{\frac{p}{p^*}} &\le S^{-1}\int_D |\nabla w_L|^p\mathrm{d}x = S^{-1}\int_D |\nabla\big(v_L^{\tau-1}v\big)|^p\mathrm{d}x\\ &= S^{-1}\int_D |(\tau-1)uv_L^{\tau-2}\nabla v_L + v_L^{\tau-1}\nabla v|^p\mathrm{d}x\\ &\le 2^{p-1}S^{-1}\left[\int_D |(\tau-1)uv_L^{\tau-2}\nabla v_L|^p + \int_D v_L^{p(\tau-1)}|\nabla v|^p\mathrm{d}x\right]\\ &= 2^{p-1}S^{-1}\left[\int_{D\cap\{v\le L\}} (\tau-1)^p v_L^{p(\tau-1)}|\nabla v|^p + \int_D v_L^{p(\tau-1)}|\nabla v|^p\mathrm{d}x\right]\\ &\le 2^{p-1}S^{-1}\tau^p\left[\left(\frac{\tau-1}{\tau}\right)^p + \frac{1}{\tau^p}\right]\int_D v_L^{p(\tau-1)}|\nabla v|^p\mathrm{d}x\\ &\le 2^pS^{-1}\tau^p\int_D v_L^{p(\tau-1)}|\nabla v|^p\mathrm{d}x, \end{aligned} \tag{3.13}$$

where S is given by

$$S = \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla v|^p\mathrm{d}x}{(\int_\Omega |v|^{p^*}\mathrm{d}x)^{\frac{p}{p^*}}}.$$

It follows from above inequality and $\|v\| \le m_0$,

$$\|v\|_{p^*} \le S^{-\frac{1}{p}}\|v\| \le m_0S^{-\frac{1}{p}}. \tag{3.14}$$

Thus

$$\begin{aligned}&\lambda K^{\beta-s}K_0^{s+1}\|v\|_{p^*}^{\frac{p^*(\alpha^*-p)}{\alpha^*}} + K_0^{1-\gamma}K^{\frac{1-\gamma}{2}-p}|\Omega|^{\frac{1}{\theta}}\|a\|_q\\ &\le \lambda K^{\beta-s}K_0^{s+1}\big(m_0S^{-\frac{1}{p}}\big)^{\frac{p^*(\alpha^*-p)}{\alpha^*}} + K_0^{1-\gamma}K^{\frac{1-\gamma}{2}-p}|\Omega|^{\frac{1}{\theta}}\|a\|_q := C_{\lambda,K}.\end{aligned}$$

From (3.12)–(3.14), we have

$$\begin{aligned}\left(\int_D |w_L|^{p^*}\mathrm{d}x\right)^{\frac{p}{p^*}} &\le 2^pS^{-1}\tau^p\int_D v_L^{p(\tau-1)}|\nabla v|^p\mathrm{d}x\\ &\le 2^pS^{-1}\tau^pC_{\lambda,K}\|w_L\|^p_{L^{\alpha^*}(D)}.\end{aligned} \tag{3.15}$$

Set $\tau := \frac{p^*}{\alpha^*}$, since $v_L \le v$, we conclude that $w_L \in L^{\alpha^*}(D)$, whenever $v^\tau \in L^{\alpha^*}(D)$. We have from (3.15) that

$$\begin{aligned}\left(\int_D v_L^{p^*(\tau-1)}v^{p^*}\mathrm{d}x\right)^{\frac{1}{p^*}} &\le 2\tau S^{-\frac{1}{p}}C_{\lambda,K}^{\frac{1}{p}}\left(\int_D v_L^{\alpha^*(\tau-1)}v^{\alpha^*}\mathrm{d}x\right)^{\frac{1}{\alpha^*}}\\ &\le 2\tau S^{-\frac{1}{p}}C_{\lambda,K}^{\frac{1}{p}}\left(\int_D v^{\alpha^*\tau}\mathrm{d}x\right)^{\frac{1}{\alpha^*}}.\end{aligned}$$

We now apply Fatou's lemma to the variable L to obtain

$$\|v\|_{L^{\tau p^*}(D)} \le 2^{\frac{1}{\tau}} S^{-\frac{1}{p\tau}} \tau^{\frac{1}{\tau}} C_{\lambda,K}^{\frac{1}{p\tau}} \|v\|_{L^{\tau\alpha^*}(D)} \tag{3.16}$$

where $v^{\tau\alpha^*} \in L^1(D)$. Since $\tau = \frac{p*}{\alpha^*} > 1$ and $v \in L^{p^*}(D)$, the inequality (3.16) holds for this choice of τ. Thus, since $\tau^2\alpha^* = \tau p^*$, it follows that (3.16) also holds with τ replaced by τ^2. Hence

$$\begin{aligned}
\|v\|_{L^{\tau^2 p^*}(D)} &\le \left(2^{\frac{1}{\tau^2}} S^{-\frac{1}{p\tau^2}} \tau^{\frac{2}{\tau^2}} C_{\lambda,K}^{\frac{1}{p\tau^2}}\right)\|v\|_{L^{\tau^2\alpha^*}(D)} \\
&\le \left(2^{\frac{1}{\tau^2}} S^{-\frac{1}{p\tau^2}} \tau^{\frac{2}{\tau^2}} C_{\lambda,K}^{\frac{1}{p\tau^2}}\right) 2^{\frac{1}{\tau}} S^{-\frac{1}{p\tau}} \tau^{\frac{1}{\tau}} C_{\lambda,K}^{\frac{1}{p\tau}} \|v\|_{L^{\tau\alpha^*}(D)} \\
&= \left(2^{\frac{1}{\tau^2}+\frac{1}{\tau}} S^{-\frac{1}{p}(\frac{1}{\tau^2}+\frac{1}{\tau})} \tau^{\frac{2}{\tau^2}+\frac{1}{\tau}} C_{\lambda,K}^{\frac{1}{p}(\frac{1}{\tau^2}+\frac{1}{\tau})}\right)\|v\|_{L^{p^*}(D)}
\end{aligned}$$

By iterating this process, we obtain

$$\|v\|_{L^{\tau^m p^*}(D)} \le 2^{\sum_{i=1}^m \tau^{-i}} S^{-\frac{1}{p}\sum_{i=1}^m \tau^{-i}} \tau^{\sum_{i=1}^m \frac{i}{\tau^i}} C_{\lambda,K}^{\frac{1}{p}\sum_{i=1}^m \tau^{-i}} \|v\|_{L^{p^*}(D)}.$$

Taking limit as $m \to \infty$, we obtain

$$\|v\|_{L^\infty(D)} \le 2^{p\sigma_1} S^{-\sigma_1} \tau^{\sigma_2} C_{\lambda,K}^{\sigma_1} m_0 S^{-\frac{1}{p}} \le C^* C_{\lambda,K}^{\sigma_1},$$

where $\sigma_1 = \frac{1}{p}\sum_{i=1}^\infty \tau^{-i}$, $\sigma_2 = \sum_{i=1}^\infty \frac{i}{\tau^i}$ and $C^* = 2^{p\sigma_1} S^{-\sigma_1}\tau^{\sigma_2} m_0 S^{-\frac{1}{p}}$.

Next, we will find some suitable value of λ and K, such that

$$C^* C_{\lambda,K}^{\sigma_1} \le K,$$

that is,

$$\lambda K^{\beta-s} K_0^{s+1} \|v\|_{p^*}^{\frac{p^*(\alpha^*-p)}{\alpha^*}} + K_0^{1-\gamma} K^{\frac{1-\gamma}{2}-p} |\Omega|^{\frac{1}{\theta}} \|a\|_q \le \left(\frac{K}{C^*}\right)^{\frac{1}{\sigma_1}}. \tag{3.17}$$

One may note that C^* is dependent on m_0 which is decreasing with respect to K, (see Remark 3.2) and $\frac{1-\gamma}{2} - p < 0$. Thus we can choose $K > 0$ large to satisfy the inequality

$$\left(\frac{K}{C^*}\right)^{\frac{1}{\sigma_1}} - K_0^{1-\gamma} K^{\frac{1-\gamma}{2}-p} |\Omega|^{\frac{1}{\theta}} \|a\|_q > 0, \tag{3.18}$$

and then fix λ_K such that

$$\lambda \le \lambda_K := \frac{1}{K^{\beta-s} K_0^{s+1} (m_0 S^{-\frac{1}{p}})^{\frac{p^*(\alpha^*-p)}{\alpha^*}}} \left[\left(\frac{K}{C^*}\right)^{\frac{1}{\sigma_1}} - K_0^{1-\gamma} K^{\frac{1-\gamma}{2}-p} |\Omega|^{\frac{1}{\theta}} \|a\|_q\right]. \tag{3.19}$$

Let $\lambda^* = \min\{\lambda_*, \lambda_K\}$. Thus, we obtain (3.17) for $\lambda \in (0, \lambda^*)$ and some fixed $K > 0$ satisfying (3.18), i.e.

$$\|v\|_{L^\infty(D)} \le K, \quad \forall \lambda \in \left(0, \lambda^*\right)$$

and by the definition of D, we have $\|u\|_{L^\infty(\Omega\setminus D)} \le K$. To summarize, we have $\|v\|_{L^\infty(\Omega)} \le K, \forall\lambda \in (0, \lambda^*)$. □

References

1. Agarwal, R.P., Perera, K., O'Regan, D.: A variational approach to singular quasilinear elliptic problems with sign changing nonlinearities. Appl. Anal. **85**, 1201–1206 (2006)
2. Ambrosetti, A., Brezis, H., Cerami, G.: Combined effects of concave and convex nonlinearities in some elliptic problems. J. Funct. Anal. **122**, 519–543 (1994)
3. Ambrosetti, A., Rabinowitz, P.H.: Dual variational methods in critical point theory and applications. J. Funct. Anal. **14**, 349–381 (1973)
4. Bass, F.G., Nasanov, N.N.: Nonlinear electromagnetic spin waves. Phys. Rep. **189**, 165–223 (1990)
5. Berestycki, H., Lions, P.-L.: Nonlinear scalar field equations. I: Existence of a ground state. Arch. Ration. Mech. Anal. **82**, 313–345 (1983)
6. Brandi, H.S., Manus, C., Mainfray, G., Lehner, T., Bonnaud, G.: Relativistic and ponderomotive self-focusing of a laser beam in a radially inhomogeneous plasma. I. Paraxial approximation. Phys. Fluids B **5**, 3539–3550 (1993)
7. Chen, X.L., Sudan, R.N.: Necessary and sufficient conditions for self-focusing of short ultraintense laser pulse. Phys. Rev. Lett. **70**, 2082–2085 (1993)
8. Chipot, M.: Elliptic Equations: An Introductory Course. Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser, Basel (2009)
9. Colin, M., Jeanjean, L.: Solutions for a quasilinear Schrödinger equation: a dual approach. Nonlinear Anal. **56**, 213–226 (2004)
10. Corrêa, F.J.S.A., Figueiredo, G.M.: On an elliptic equation of p-Kirchhoff type via variational methods. Bull. Aust. Math. Soc. **74**, 263–277 (2006)
11. Drábek, P., Hernández, J.: Existence and uniqueness of positive solutions for some quasilinear elliptic problems. Nonlinear Anal. **44**, 189–204 (2001)
12. do Ó, J.M.B., Miyagaki, O.H., Soares, S.H.M.: Soliton solutions for quasilinear Schrödinger equations with critical growth. J. Differ. Equ. **248**, 722–744 (2010)
13. Deng, Y., Peng, S., Yan, S.: Positive soliton solutions for generalized quasilinear Schrödinger equations with critical growth. J. Differ. Equ. **258**, 115–147 (2015)
14. Fang, X.-D., Szulkin, A.: Mutiple solutions for a quasilinear Schrödinger equation. J. Differ. Equ. **254**, 2015–2032 (2013)
15. García Azorero, J.P., Peral Alonso, I., Manfredi, J.J.: Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations. Commun. Contemp. Math. **2**, 385–404 (2000)
16. Gasiński, L., Papageorgiou, N.S.: Nonlinear Analysis. Series in Mathematical Analysis and Applications, vol. 9. Chapman Hall/CRC, Boca Raton (2006)
17. Gasiński, L., Papageorgiou, N.S.: Nonlinear elliptic equations with singular terms and combined nonlinearities. Ann. Henri Poincaré **13**, 481–512 (2012)
18. Giacomoni, J., Schindler, I., Takáč, P.: Sobolev versus Hölder local minimizers and existence of multiple solutions for a singular quasilinear equation. Ann. Sc. Norm. Super. Pisa, Cl. Sci. **6**, 117–158 (2007)
19. Hasse, R.W.: A general method for the solution of nonlinear soliton and kink Schrödinger equations. Z. Phys. B **37**, 83–87 (1980)
20. Litvak, A.G., Sergeev, A.M.: One dimensional collapse of plasma waves. JETP Lett. **27**, 517–520 (1978)
21. Liu, D.: Soliton solutions for a quasilinear Schrödinger equation. Electron. J. Differ. Equ. **2013**, 267 (2013)
22. Liu, X.-Q., Liu, J.-Q., Wang, Z.-Q.: Quasilinear elliptic equations via perturbation method. Proc. Am. Math. Soc. **141**, 253–263 (2013)
23. Liu, X.-Q., Liu, J.-Q., Wang, Z.-Q.: Quasilinear elliptic equations with critical growth via perturbation method. J. Differ. Equ. **254**, 102–124 (2013)
24. Liu, J., Wang, Z.-Q.: Soliton solutions for quasilinear Schrödinger equations. I. Proc. Am. Math. Soc. **131**, 441–448 (2003)
25. Liu, J.-Q., Wang, Z.-Q., Guo, Y.-X.: Multibump solutions for quasilinear Schrödinger equations. J. Funct. Anal. **262**, 4040–4102 (2012)
26. Liu, J.-Q., Wang, Y., Wang, Z.-Q.: Soliton solutions for quasilinear Schrödinger equations. II. J. Differ. Equ. **187**, 473–493 (2003)
27. Liu, J.-Q., Wang, Y., Wang, Z.-Q.: Solutions for quasilinear Schrödinger equations via the Nehari method. Commun. Partial Differ. Equ. **29**, 879–901 (2004)
28. Liu, D., Zhao, P.: Soliton solutions for a quasilinear Schrödinger equation via Morse theory. Proc. Indian Acad. Sci. Math. Sci. **125**(3), 307–321 (2015)
29. Makhandov, V.G., Fedyanin, V.K.: Non-linear effects in quasi-one-dimensional models of condensed matter theory. Phys. Rep. **104**, 1–86 (1984)
30. Miyajima, S., Motreanu, D., Tanaka, M.: Multiple existence results of solutions for the Neumann problems via super- and sub-solutions. J. Funct. Anal. **262**, 1921–1953 (2012)

31. Moameni, A.: Existence of soliton solutions for a quasilinear Schrödinger equation involving critical exponent in $\mathbb{R}^N$. J. Differ. Equ. **229**, 570–587 (2006)
32. Perera, K., Zhang, Z.: Multiple positive solutions of singular p-Laplacian problems by variational methods. Bound. Value Probl. **3**, 377–382 (2005)
33. Poppenberg, M., Schmitt, K., Wang, Z.-Q.: On the existence of soliton solutions to quasilinear Schrödinger equations. Calc. Var. Partial Differ. Equ. **14**, 329–344 (2002)
34. Sun, Y., Wu, S., Long, Y.: Combined effects of singular and superlinear nonlinearities in some singular boundary value problems. J. Differ. Equ. **176**, 511–531 (2001)
35. Wang, X., Zhao, L., Zhao, P.: Combined effects of singular and critical nonlinearities in elliptic problems. Nonlinear Anal. **87**, 1–10 (2013)
36. Willem, M.: Minimax Theorems. Birkhäuser, Boston (1996)
37. Winkert, P.: $L^\infty(\Omega)$-estimates for nonlinear elliptic Neumann boundary value problems. Nonlinear Differ. Equ. Appl. **17**, 289–302 (2010)
38. Yang, H.: Multiplicity and asymptotic behavior of positive solutions for a singular semilinear elliptic problem. J. Differ. Equ. **189**, 487–512 (2003)
39. Zhao, L., He, Y., Zhao, P.: The existence of three positive solutions of a singular p-Laplacian problem. Nonlinear Anal. **74**, 5745–5753 (2011)
40. Zhang, Z.: Critical points and positive solutions of singular elliptic boundary value problems. J. Math. Anal. Appl. **302**, 476–483 (2005)
41. Zhang, J., Tang, X., Zhang, W.: Existence of infinitely many solutions for a quasilinear elliptic equation. Appl. Math. Lett. **37**, 131–135 (2014)
42. Zhao, L., Zhao, P.: The existence of threee solutions for p-Laplacian problems with critical and supercritical growth. Rocky Mt. J. Math. **44**, 1383–1397 (2014)